CANONICAL TRANSFORMATION

Step-1

In this, (q_i, p_i) are the canonical variables. If we can find a function $K(Q_i, P_i, t)$ such that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Where K is known as "Kamiltonian."

Step-2

Then the transformation

$$q_i \to Q_i, \quad p_i \to P_i \quad \text{where } Q_i = Q_i(q_i, p_i, t)$$
$$\& \quad P_i = P_i(q_i, p_i, t)$$

is called canonical transformation.

Step-3

We have

$$\delta \int_{t_0}^{t_1} \left(\sum P_i \dot{Q}_i - K\right) dt = 0$$

and

$$\delta \int_{t_0}^{t_1} \left(\sum p_i \dot{q}_i - H\right) dt = 0$$

We have

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{dF}{dt}$$

F is called generating function of Canonical transformation

F may have following four types :-

1) $F_1(q_i, Q_i, t)$; q_i and Q_i are treated independent.

2) $F_2(q_i, P_i, t)$; q_i and P_i " " "

3) $F_3(p_i, Q_i, t)$; p_i and Q_i " " "

4) $F_4(p_i, P_i, t)$; p_i and P_i " " "

Case-I $F = F_1(q_i, Q_i, t)$

We must have $\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{dF_1}{dt}$

$$\frac{dF_1}{dt} = \frac{d\,F_1(q_i, Q_i, t)}{dt} = \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$$

Thus we have

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$$

On Comparing the coefficients of $\dot{q}_i$ and $\dot{Q}_i$ to both sides,

We have

$$p_i = \frac{\partial F_1}{\partial q_i} \quad \text{—} (1) \qquad P_i = -\frac{\partial F_1}{\partial Q_i} \quad \text{—} (2)$$

$$K = H + \frac{\partial F_1}{\partial t} \quad \text{—} (3)$$

We see that equation (1) gives us

$$Q_i = Q_i (q_i, p_i, t)$$

and equation (2) gives

$$P_i = P_i (q_i, p_i, t)$$

Equation (3) provides the connection between the old and new Hamiltonian.

Case-II

$$F = F_2 (q_i, P_i, t)$$

$F_2 (q_i, P_i, t)$ to be obtained from $F_1 (q_i, Q_i, t)$ with

$$P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$F_2 (q_i, P_i, t) = F_1 (q_i, Q_i, t) + \Sigma P_i Q_i$$

Thus we have

$$\Sigma p_i \dot{q}_i - H = \Sigma P_i \dot{Q}_i - k + \frac{dF_1}{dt}$$

$$\Rightarrow \Sigma p_i \dot{q}_i - H = \Sigma P_i \dot{Q}_i - k + \frac{d}{dt}\left(F_2(q_i, P_i, t) - \Sigma P_i Q_i\right)$$

$$\Rightarrow \Sigma p_i \dot{q}_i - H = \cancel{\Sigma P_i \dot{Q}_i} - k + \Sigma \frac{\partial F_2}{\partial q_i} \dot{q}_i + \Sigma \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} - \cancel{\Sigma P_i \dot{Q}_i} - \Sigma Q_i \dot{P}_i$$

On Comparing Coefficients, we have

$$p_i = \frac{\partial F_2}{\partial q_i} \quad , \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$K = H + \frac{\partial F_2}{\partial t}$$

Case-III

$$F_3 = F_3(p_i, Q_i, t)$$

We want to change $F_1(q_i, Q_i, t)$ to $F_3(p_i, Q_i, t)$

through $p_i = \frac{\partial F_1}{\partial q_i}$

$$F_3(p_i, Q_i, t) = F_1(q_i, Q_i, t) - \Sigma p_i q_i$$

So, we have

$$\Sigma p_i \dot{q}_i - H = \Sigma P_i \dot{Q}_i - K + \frac{d}{dt} F_1$$

$$= \Sigma P_i \dot{Q}_i - K + \frac{d}{dt}\left(F_3(p_i, Q_i, t) + \Sigma p_i q_i\right)$$

$$\Rightarrow \Sigma p_i \dot{q}_i - H = \Sigma P_i \dot{Q}_i - K + \Sigma \frac{\partial F_3}{\partial p_i} \dot{p}_i + \Sigma \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t} + \Sigma p_i \dot{q}_i + \Sigma \dot{p}_i q_i$$

On Comparing both sides, we have

$$K = H + \frac{\partial F_3}{\partial t} \quad , \quad q_i = -\frac{\partial F_3}{\partial p_i} \quad , \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

Case-IV When $F_4 = F_4(p_i, P_i, t)$

Solving, We have

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad , \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

$$K = H + \frac{\partial F_4}{\partial t}$$

EXAMPLE Show that transformation

$$Q = \log\left(\frac{1}{q}\sin p\right) \quad , \quad P = q\cot p$$

is canonical.

Sol^n We have for (q, p) $\qquad (q,p) \to (Q,P)$

$Q = Q(q, p, t)$

$P = P(q, p, t)$

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

We should have

$$\dot{Q} = \frac{\partial K}{\partial P} \quad , \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

To show that

IF $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$ hold

$$\dot{Q} = \frac{\partial H}{\partial P} \quad , \quad \dot{P} = -\frac{\partial H}{\partial Q}$$

$$\delta\left(\int_{t_0}^{t_1} L\,dt\right) = 0$$

Now

$$\dot{Q} = \frac{d}{dt}Q$$

$$\delta\int_{t_0}^{t_1}\left(L + \frac{dF}{dt}\right)dt = 0$$

$$= \frac{d}{dt}\left(\log\frac{1}{q}\sin p\right)$$

$$= \frac{q}{\sin p}\frac{d}{dt}\left(\frac{1}{q}\sin p\right)$$

$$K = H + \frac{\partial F}{\partial t}$$

classmate

Date
Page

$$= \frac{q}{\sin p}\left(\frac{q\cos p\,\dot{p} - \dot{q}\sin p}{q^2}\right)$$

$$= \frac{1}{q\sin p}\left(q\cos p\,\dot{p} - \dot{q}\sin p\right)$$

$$= \frac{1}{q\sin p}\left(q\cos p\left(-\frac{\partial H}{\partial q}\right) - \frac{\partial H}{\partial p}\sin p\right)$$

$$= \frac{-1}{q\sin p}\left[q\cos p\,\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\sin p\right]$$

$$= \frac{-1}{q\sin p}\frac{\partial H}{\partial P}\left[q\cos p\cot p + (-q\,\mathrm{cosec}^2 p)\sin p\right]$$

$$= \frac{-1}{q\sin p}\frac{\partial H}{\partial P}\left[\frac{q\cos^2 p}{\sin p} - \frac{q}{\sin p}\right]$$

$$= \frac{-1}{\sin p}\frac{\partial H}{\partial P}\left(\frac{\cos^2 p - 1}{\sin p}\right)$$

$$= \frac{1}{\sin p}\frac{\partial H}{\partial P}\left(\frac{1-\cos^2 p}{\sin p}\right)$$

$$= \frac{1}{\sin p}\frac{\partial H}{\partial P}\left(\frac{\sin^2 p}{\sin p}\right)$$

$$= \frac{\partial H}{\partial P}$$

Now $\dot{P} = \frac{d}{dt}P = \frac{d}{dt}(q\cot p)$

$$= -q\,\mathrm{cosec}^2 p\,\dot{p} + \cot p\,\dot{q}$$

Scanned with CamScanner

$$= \frac{\partial H}{\partial p} \cot p + q \operatorname{cosec}^2 p \frac{\partial H}{\partial q}$$

$$= \frac{\partial H}{\partial Q} \frac{\partial Q}{\partial p} \cot p + q \operatorname{cosec}^2 p \frac{\partial H}{\partial Q} \frac{\partial Q}{\partial q}$$

$$= \frac{\partial H}{\partial Q} \left[\cot p \frac{\partial Q}{\partial p} + q \operatorname{cosec}^2 p \frac{\partial Q}{\partial q} \right]$$

$$= \frac{\partial H}{\partial Q} \left[\cot p \frac{q}{\sin p} \cdot \left(\frac{1}{q} \cos p \right) + q \operatorname{cosec}^2 p \cdot \frac{q}{\sin p} \left(\frac{-1}{q^2} \right) \sin p \right]$$

$$= \frac{\partial H}{\partial Q} \left[\frac{\cos^2 p}{\sin^2 p} - \frac{1}{\sin^2 p} \right]$$

$$= \frac{\partial H}{\partial Q} \left[\frac{\cos^2 p - 1}{\sin^2 p} \right]$$

$$= \frac{\partial H}{\partial Q} \left[\frac{-\sin^2 p}{\sin^2 p} \right]$$

$$= \frac{-\partial H}{\partial Q}$$

2nd method

From generating function $F_1(q, Q)$ when F_1 does not involve t, we have

$$p = \frac{\partial F_1}{\partial q} \quad , \quad P = \frac{-\partial F_1}{\partial Q}$$

$$F_1 = F_1(q, Q)$$

$$dF_1 = \frac{\partial F_1}{\partial q} dq + \frac{\partial F_1}{\partial Q} dQ$$

$$\Rightarrow dF_1 = p\,dq - P\,dQ$$

The aliter way to prove the given transfermotion to be canonical is that one should prove that the quantity

$p\,dq - P\,dQ$ is the total differential.

Now $p\,dq - P\,dQ = p\,dq - (q\cot p)\, d\left[\log\left(\frac{\sin p}{q}\right)\right]$

$$= p\,dq - (q\cot p)\left(\frac{q}{\sin p}\right)\left[\frac{q\cos p\,dp - \sin p\,dq}{q^2}\right]$$

$$= p\,dq - \frac{\cos p}{\sin^2 p}\left(q\cos p\,dp - \sin p\,dq\right)$$

$$= p\,dq - \frac{q\cos^2 p}{\sin^2 p}\,dp + \frac{\cos p}{\sin p}\,dq$$

$$= p\,dq - (\cot^2 p)\,q\,dp + \cot p\,dq$$

$$= (p + \cot p)\,dq - q\cot^2 p\,dp$$

$$= d(pq + q\cot p)$$ H.P.

$$\left[\begin{aligned} \because d(pq + q\cot p) &= p\,dq + q\,dp - q\,\mathrm{cosec}^2 p\,dp + \cot p\,dq \\ &= (p + \cot p)\,dq - q(\mathrm{cosec}^2 p - 1)\,dp \\ &= (p + \cot p)\,dq - q\cot^2 p\,dp. \end{aligned}\right]$$

classmate

Date ____
Page ____

Questions From Exercise of Book "Mechanics" By Goldstien.

Exp ① $Q = \log(1+\sqrt{q}\cos p)$

P.T. transformations are canonical.

$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$

② For what values of α & β, the transformation

$$Q = q^{\alpha}\cos\beta p$$

and $P = q^{\alpha}\sin\beta p$ are canonical.

Proof ① I^st Method

$$\dot{Q} = \frac{d}{dt}Q$$

$$= \frac{1}{1+\sqrt{q}\cos p}\frac{d}{dt}(1+\sqrt{q}\cos p)$$

$$= \frac{1}{1+\sqrt{q}\cos p}\left[\frac{1}{2\sqrt{q}}\cos p\,\dot{q} - \sqrt{q}\sin p\,\dot{p}\right]$$

$$= \frac{1}{1+\sqrt{q}\cos p}\left[\frac{1}{2\sqrt{q}}\cos p\frac{\partial H}{\partial p} + \sqrt{q}\sin p\frac{\partial H}{\partial q}\right]$$

$$= \frac{1}{1+\sqrt{q}\cos p}\left[\frac{1}{2\sqrt{q}}\cos p\frac{\partial H}{\partial P}\frac{\partial P}{\partial p} + \sqrt{q}\sin p\frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right]$$

$$= \frac{\partial H}{\partial P}\frac{1}{1+\sqrt{q}\cos p}\left[\frac{1}{2\sqrt{q}}\cos p\frac{\partial P}{\partial p} + \sqrt{q}\sin p\frac{\partial P}{\partial q}\right]$$

classmate

Date

Page

$$= \frac{\partial H}{\partial p} \frac{1}{1+\sqrt{q}\cos p} \Big[\frac{1}{\cancel{2\sqrt{q}}} \cos p \quad \cancel{2\sqrt{q}} \sin p \left(-\sqrt{q} \sin p\right)$$

$$+ \frac{\cos p}{\cancel{2\sqrt{q}}} \left(1+\sqrt{q}\cos p\right) \left(\cancel{2\sqrt{q}} \cos p\right)$$

$$+ \cancel{\sqrt{q}} \sin p \; \cancel{2} \left(1+\sqrt{q}\cos p\right) \frac{1}{\cancel{2\sqrt{q}}} \sin p$$

$$+ \cancel{\sqrt{q}} \sin p \; \cancel{2} \left(\frac{1}{\cancel{2\sqrt{q}}} \cos p\right) \sqrt{q} \sin p \Big]$$

$$= \frac{\partial H}{\partial p} \frac{1}{1+\sqrt{q}\cos p} \Big[\cancel{-\sqrt{q}\sin^2 p \cos p} + \cos^2 p$$

$$+ \sqrt{q}\cos^3 p + \sin^2 p + \cancel{\sqrt{q}\sin^2 p \cos p}$$

$$+ \sqrt{q} \sin^2 p \cos p \Big]$$

$$= \frac{\partial H}{\partial p} \frac{1}{1+\sqrt{q}\cos p} \left[\left(\cos^2 p + \sin^2 p\right) + \sqrt{q}\cos p \left(\cos^2 p + \sin^2 p\right) \right]$$

$$= \frac{\partial H}{\partial p} \frac{1}{1+\sqrt{q}\cos p} \left[1 + \sqrt{q} \cos p \right]$$

$$= \frac{\partial H}{\partial p}$$

Now $\dot{p} = \frac{d}{dt} p$

$$= \frac{d}{dt} \left[2 \left(1+\sqrt{q}\cos p\right) \sqrt{q} \sin p \right]$$

Date
Page

$$= \cancel{2}\frac{1}{\cancel{2}\sqrt{q}} \sin p \left(1+\sqrt{q}\cos p\right)\dot{q}$$

$$+ 2\sqrt{q}\cos p \left(1+\sqrt{q}\cos p\right)\dot{p}$$

$$+ 2\sqrt{q}\sin p \left(0 + \frac{1}{2\sqrt{q}}\dot{q}\cos p - \sqrt{q}\sin p\,\dot{p}\right)$$

$$= \left[\frac{1}{\sqrt{q}}\sin p\left(1+\sqrt{q}\cos p\right) + \sin p\cos p\right]\dot{q}$$

$$+ \left[2\sqrt{q}\cos p\left(1+\sqrt{q}\cos p\right) - 2q\sin^2 p\right]\dot{p}$$

$$= \left[\frac{1}{\sqrt{q}}\sin p + \sin p\cos p + \sin p\cos p\right]\dot{q}$$

$$+ \left[2\sqrt{q}\cos p + 2q\cos^2 p - 2q\sin^2 p\right]\dot{p}$$

$$= \left[\frac{1}{\sqrt{q}}\sin p + 2\sin p\cos p\right]\frac{\partial H}{\partial p}$$

$$- \left[2\sqrt{q}\cos p + 2q\cos 2p\right]\frac{\partial H}{\partial q}$$

$$= \frac{\partial H}{\partial Q}\left\{\left[\frac{\sin p}{\sqrt{q}} + \sin 2p\right]\frac{\partial Q}{\partial p} - \left[2\sqrt{q}\cos p + 2q\cos 2p\right]\frac{\partial Q}{\partial q}\right\}$$

$$= \frac{\partial H}{\partial Q}\left[\left(\frac{\sin p}{\sqrt{q}} + \sin 2p\right)\left(\frac{-\sqrt{q}\sin p}{1+\sqrt{q}\cos p}\right) - \left(2\sqrt{q}\cos p + 2q\cos 2p\right)\left(\frac{\cos p}{1+\sqrt{q}\cos p}\right)\left(\frac{1}{2\sqrt{q}}\right)\right]$$

$$= \frac{-\partial H}{\partial Q}\frac{1}{1+\sqrt{q}\cos p}\left[\sin^2 p - \sqrt{q}\sin p \sin 2p + \cos^2 p + \sqrt{q}\cos 2p\cos p\right]$$

$$= \frac{-\partial H}{\partial Q}\frac{1}{1+\sqrt{q}\cos p}\left[(\sin^2 p + \cos^2 p) + \sqrt{q}\left(\cos^3 p - \sin^2 p\cos p + 2\sin^2 p\cos p\right)\right]$$

$$= \frac{-\partial H}{\partial Q}\frac{1}{1+\sqrt{q}\cos p}\left[1 + \sqrt{q}\cos p\left(\cos^2 p + \sin^2 p\right)\right]$$

$$= \frac{-\partial H}{\partial Q}\frac{1}{\cancel{1+\sqrt{q}\cos p}}\left[\cancel{1+\sqrt{q}\cos p}\right]$$

$$= -\frac{\partial H}{\partial Q}$$

Hence, transformations are canonical.

2nd Method

From generating function $F_1(q, Q)$, where F_1 does not involve t, we have

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}$$

$$F_1 = F_1(q, Q)$$

$$dF_1 = \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial Q}dQ$$

$\Rightarrow \; dF_1 = p\,dq - P\,dQ$

So, to prove the given transformation to be canonical is that one should prove the quantity $p\,dq - P\,dQ$ is the total differential.

Now $p\,dq - P\,dQ$

$$= p\,dq - 2\,\cancel{(1+\sqrt{q}\cos p)}\sqrt{q}\sin p \times \frac{1}{\cancel{(1+\sqrt{q}\cos p)}}\, d(1+\sqrt{q}\cos p)$$

$$= p\,dq - 2\sqrt{q}\sin p\left[-\sqrt{q}\sin p\,dp + \frac{1}{2\sqrt{q}}\cos p\,dq\right]$$

$$= \left(p\,dq - \sin p\cos p\,dq\right) + 2q\sin^2 p\,dp$$

$$= \underbrace{(p - \sin p\cos p)}_{M}\,dq + \underbrace{2q\sin^2 p}_{N}\,dp$$

Now

$$\frac{\partial M}{\partial p} = 1 - \left(-\sin p.\sin p + \cos^2 p\right)$$

$$= 1 - \cos 2p$$

$$= 2\sin^2 p$$

$$\frac{\partial N}{\partial q} = 2\sin^2 p$$

$$\therefore \; \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

$\therefore$ Given diff eq^n is exact.

$$\therefore p\,dq - P\,dQ = d\left(\int_{p\ \text{Constant}} M\,dq + \int N\,dp\right)$$

Terms not containing q

$$= d\left(\int_{p\ \text{Constant}} (p - \sin p \cos p)\,dq + 0\right)$$

$$= d\left((p - \sin p \cos p)q\right)$$

Hence, ~~Canonical~~ these are canonical transformations.

---

(2)

As, $Q = q^{\alpha} \cos\beta p$, $P = q^{\alpha} \sin\beta p$

Now $p\,dq - P\,dQ$

$$= p\,dq - (q^{\alpha}\sin\beta p)\,d(q^{\alpha}\cos\beta p)$$

$$= p\,dq - (q^{\alpha}\sin\beta p)(\alpha q^{\alpha-1}\cos\beta p\,dq - q^{\alpha}\beta\sin\beta p\,dp)$$

$$= p\,dq - \alpha q^{2\alpha-1}\sin\beta p\cos\beta p\,dq + q^{2\alpha}\beta\sin^2\beta p\,dp$$

$$= \underbrace{\left(p - \alpha q^{2\alpha-1}\frac{\sin 2\beta p}{2}\right)}_{M} dq + \underbrace{\left(q^{2\alpha}\beta\sin^2\beta p\right)}_{N} dp$$

Now, transformations are canonical
if $p\,dq - P\,dQ$ is total differential
if diff eq^n is exact

if $\dfrac{\partial M}{\partial p} = \dfrac{\partial N}{\partial q}$

Now $\frac{\partial M}{\partial p} = 1 - \frac{\alpha q^{2\alpha-1}}{2} \frac{\partial}{\partial p}(\sin 2\beta p)$

$= 1 - \frac{1}{2}\alpha q^{2\alpha-1}(2\beta \cos 2\beta p)$

$= 1 - (\alpha\beta q^{2\alpha-1}) \cos 2\beta p$

& $\frac{\partial N}{\partial q} = 2\alpha\beta\, q^{2\alpha-1} \sin^2 \beta p.$

Put $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial p}$

$\Rightarrow 1 - (\alpha\beta q^{2\alpha-1}) \cos 2\beta p = (\alpha\beta q^{2\alpha-1})(2\sin^2\beta p)$

To get above equality,

Put $\alpha\beta q^{2\alpha-1} = 1$

$\Rightarrow 2\alpha - 1 = 0$

$\Rightarrow \alpha = \frac{1}{2}$

$\therefore \alpha\beta\, q^{2\alpha-1} = 1$

$\frac{1}{2}\beta\, q^{0} = 1$

$\Rightarrow \beta = 2.$

$\therefore \alpha = \frac{1}{2}, \quad \beta = 2$

ANSWER

Ques The transformation equations between two sets of co-ordinates are

$Q = \log(1 + q^{1/2}\cos p)$ & $P = 2(1 + q^{1/2}\cos p) q^{1/2} \sin p$

Show that function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p.$$

Proof Step-1 As we know,

$$dF_1 = p\,dq - P\,dQ \quad \text{— ①}$$

and $F_3 = F_1 - pq \quad$ — ②

Step-2

From ①,

$$dF_1 = p\,dq - 2\cancel{(1+\sqrt{q}\cos p)}\sqrt{q}\sin p \times \frac{1}{\cancel{(1+\sqrt{q}\cos p)}} \times d(1+\sqrt{q}\cos p)$$

$$\Rightarrow dF_1 = p\,dq - 2\sqrt{q}\sin p\left[-\sqrt{q}\sin p\,dp + \frac{1}{2\sqrt{q}}\cos p\,dq\right]$$

$$= (p\,dq - \sin p\cos p\,dq) + 2q\sin^2 p\,dp$$

$$= (p - \sin p\cos p)\,dq + (2q\sin^2 p)\,dp$$

$$= d\left((p - \sin p\cos p)q\right)$$

$$= d\left((p - \sin p\cos p)q\right)$$

$$\Rightarrow F_1 = pq - q\sin p\cos p$$

Step-3 From ②, $F_3 = F_1 - pq$

$$= -q\sin p\cos p$$

$$\therefore \quad F_3 = -q \sin p \cos p$$

$$= -q \frac{\sin p}{\cos p} \cos^2 p$$

$$= -\left(q^{1/2} \cos p\right)^2 \tan p$$

$$= -\left[\left(1 + q^{1/2} \cos p\right) - 1\right]^2 \tan p$$

$$= -\left(e^{Q} - 1\right)^2 \tan p$$

$$\left[\begin{array}{l} \because \; Q = \log\left(1 + q^{1/2} \cos p\right) \\ \Rightarrow e^{Q} = \left(1 + q^{1/2} \cos p\right) \end{array}\right]$$

$$\therefore \; F_3(p, Q) = -\left(e^{Q} - 1\right)^2 \tan p$$

Hence Proved

Alternate

We are given the following generating function of the F_3 type:

$$F_3 = -\left(e^{Q} - 1\right)^2 \tan p$$

For a generating function of F_3 type, q is given as

$$q = -\frac{\partial F_3}{\partial p}$$

$$\Rightarrow \quad q = \left(e^{Q} - 1\right)^2 \sec^2 p$$

$$\Rightarrow \quad \sqrt{q} \cos p = e^{Q} - 1$$

$$\Rightarrow \quad \boxed{Q = \log\left(1 + \sqrt{q} \cos p\right)} \quad \text{--- (1)}$$

Date ______
Page ______

and P is given as

$$P = -\frac{\partial F_3}{\partial Q}$$

$$\Rightarrow P = 2\tan p\,(e^{Q} - 1)\,e^{Q}$$

Putting value of Q from (1) in above eq^n,

$$\Rightarrow P = \frac{2\sin p}{\cos p}\left(1 + \sqrt{q}\cos p - 1\right)\left(1 + \sqrt{q}\cos p\right)$$

$$= \frac{2\sin p}{\cancel{\cos p}}\sqrt{q}\,\cancel{\cos p}\left(1 + \sqrt{q}\cos p\right)$$

$$= 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p$$

Therefore the given generating function does in fact generate the given transformation.

Hence Proved

Ques Find under what conditions $Q = \frac{\alpha p}{x}$, $P = \beta x^2$ where α & β are constants, represents a canonical transformation for a system of one degree of freedom, & obtain a suitable generating function.

Proof. From generating function $F_1(q, Q)$, where F_1 does not involve t, we have

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}$$

$$F_1 = F_1(q, Q)$$

$$dF_1 = \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial Q}dQ$$

$$dF_1 = p\,dq - P\,dQ$$

So, to prove the given transformation to be canonical is that one should prove the quantity $p\,dq - P\,dQ$ is the total differential.

Now Replacing $x \leftrightarrow q$, we get $Q = \frac{\alpha p}{q}$, $P = \beta q^2$

$$p\,dq - P\,dQ$$

$$= p\,dq - (\beta q^{2})\,\alpha\left(\frac{q\,dp - p\,dq}{q^2}\right)$$

$$= p\,dq - \alpha\beta\,(q\,dp - p\,dq)$$

$$= (p\,dq + \alpha\beta p\,dq) - \alpha\beta q\,dp$$

$$= \underbrace{p(1+\alpha\beta)}_{M}\,dq + \underbrace{(-\alpha\beta q)}_{N}\,dp$$

$$\frac{\partial M}{\partial p} = 1+\alpha\beta \quad , \quad \frac{\partial N}{\partial q} = -\alpha\beta.$$

Put $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

$$\Rightarrow 1+\alpha\beta = -\alpha\beta$$

$$\Rightarrow 2\alpha\beta = -1$$

$$\Rightarrow \beta = \frac{-1}{2\alpha}$$ is the reqd. condition.

Now $p\,dp - P\,dQ = p(1+\alpha\beta)\,dq + (-\alpha\beta q)\,dp$

$$= p\left(1-\frac{1}{2}\right)dq + \frac{1}{2}q\,dp$$

$$= d\left(\frac{1}{2}pq\right) = dF_1$$

$\therefore\ F_1 = \frac{1}{2}pq = \frac{1}{2}q\left(\frac{qQ}{\alpha}\right)$ — $\boxed{\because Q = \frac{\alpha p}{q} \Rightarrow p = \frac{Qq}{\alpha}}$

$\Rightarrow F_1(q, Q) = \frac{1}{2\alpha}q^2 Q$

is the required generating function.